Computer Mathematics

Week 2
Positional number systems

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last week

what is an electronic, digital computer?

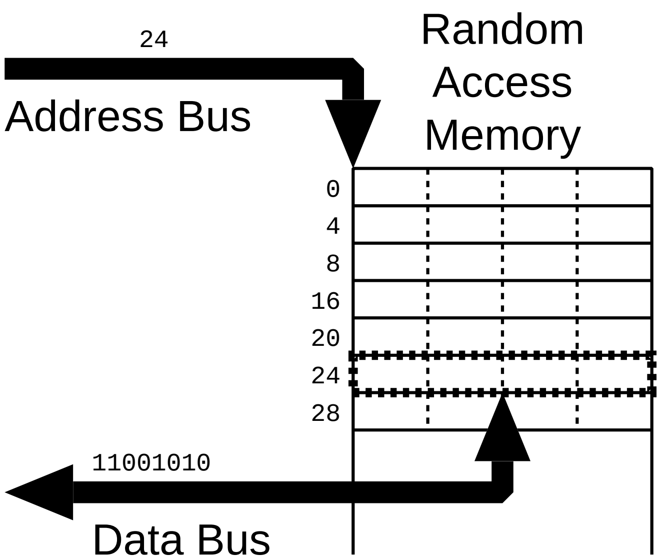
- stores, manipulates, communicates binary data

binary data

- bits, bytes, words

stores data: Random(ly) Access(ible) Memory

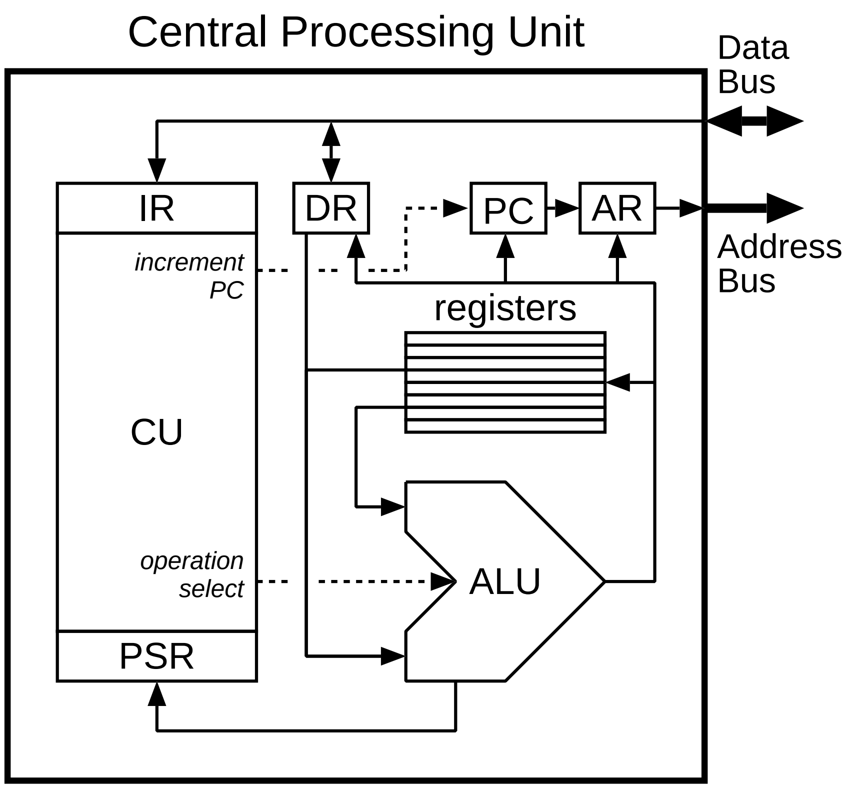
- each byte (or word) has a unique numeric address
  - from 0 to $sizeOfMemory - 1$
- supplied via the address bus
- data can be written/read via the data bus



last week

manipulates data: Central Processing Unit

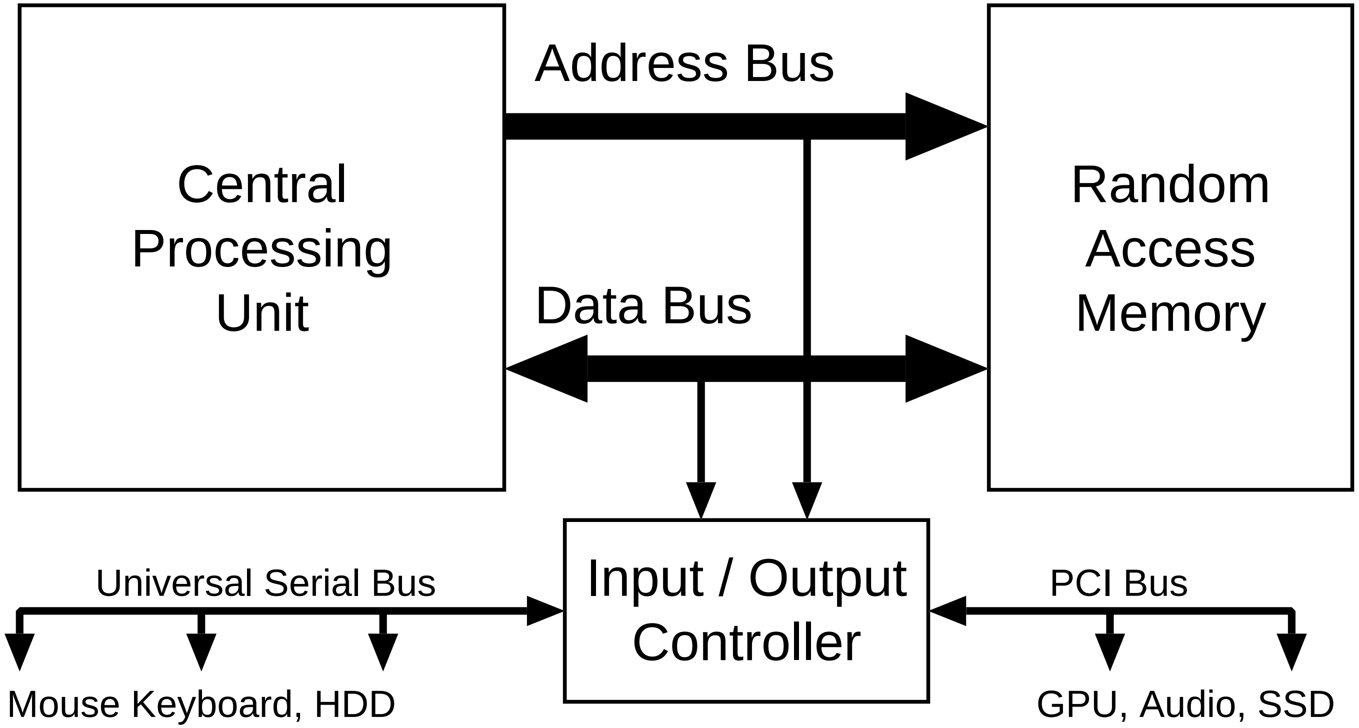
- Program Counter
- Address Register
- Data Register
- Instruction Register
- Control Unit
- General Purpose Registers
- Arithmetic and Logic Unit
- Processor Status Register



last week

Communicates data: Input/Output Controller

- PCI — Peripheral Component Interconnect
- USB — Universal Serial Bus



positional number representations

each *digit position* in a number has a different *significance* or *weight*

each weight is related to the next one by a constant multiplier

- the multiplier is called the *base* or *radix* r of the number
- position 0, immediately before the *radix point*, has weight 1 (r^0)
- as you move left, each weight is multiplied by the radix
- as you move right, each weight is divided by the radix

$$r^2 \quad r^1 \quad r^0 \quad . \quad r^{-1} \quad r^{-2}$$

↑

radix point

(or *'separator'*)

the value of the digit d_i in position i is

- the digit multiplied by the weight of its position, $d_i \times r^i$

the value of a number is the sum of the values of all the digits

$$\sum d_i \times r^i$$

decimal system

decimal numbers have base (radix) $r = 10$

- in Latin, "decima" means "one tenth part"

there are 10 symbols for the *digits*

- $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9

position weights are powers of 10

$\ldots$	10^3	10^2	10^1	10^0	**.**	10^{-1}	10^{-2}	10^{-3}	$\ldots$
	1000	100	10	1	$\uparrow$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	

decimal point

the decimal radix is implied (or denoted by the subscript 10 when ambiguous)

e.g., converting the decimal number 36.25_{10} to decimal:

3		6	.	2		5		
10^1		10^0		10^{-1}		10^{-2}		
3×10	$+$	6×1	$+$	2×0.1	$+$	5×0.01	$= 36.25$	

binary system

binary numbers have radix $r = 2$

- in Latin, "bini" means "two together"

there are 2 symbols for the digits

- 0 and 1

position weights are powers of 2

$$\begin{array}{ccccccccc} \ldots & 2^3 & 2^2 & 2^1 & 2^0 & . & 2^{-1} & 2^{-2} & 2^{-3} & \ldots \\ & 8 & 4 & 2 & 1 & & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \\ & & & & & \uparrow & & & & \end{array}$$

binary point

the binary radix is denoted by the subscript 2

e.g., converting the binary number 100010.01_2 to decimal:

$$\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & . & 0 & 1 & \\ 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 & & 2^{-1} & 2^{-2} & \end{array}$$

$$1 \times 32 \; + \; 0 \times 16 \; + \; 0 \times 8 \; + \; 0 \times 4 \; + \; 1 \times 2 \; + \; 0 \times 1 \; + \; 0 \times \tfrac{1}{2} \; + \; 1 \times \tfrac{1}{4} \quad = 36.25$$

octal system

octal numbers have radix $r = 8$

- in Latin, “octo” means eight

8 symbols are used for the digits

- $0, 1, 2, 3, 4, 5, 6$ and 7

position weights are powers of 8

$$\begin{array}{ccccccccccc} \ldots & 8^3 & 8^2 & 8^1 & 8^0 & . & 8^{-1} & 8^{-2} & 8^{-3} & \ldots \\ & 512 & 64 & 8 & 1 & & \frac{1}{8} & \frac{1}{64} & \frac{1}{512} & \\ & & & & & \uparrow & & & & \end{array}$$

octal point

the octal radix is denoted by the subscript 8

e.g., converting the octal number 44.2_8 to decimal:

$$\begin{array}{ccccccc} 4 & & 4 & . & 2 & & \\ 8^1 & & 8^0 & & 8^{-1} & & \\ 4 \times 8 & + & 4 \times 1 & + & 2 \times \frac{1}{8} & = 36.25 \end{array}$$

hexadecimal system

hexadecimal numbers have radix $r = 16$

- in Greek, "hexa" means six (plus Latin "decima" meaning tenth)

16 symbols are used for the digits

- $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$ and F

position weights are powers of 16

$\ldots$	16^3	16^2	16^1	16^0	**.**	16^{-1}	16^{-2}	16^{-3}	$\ldots$
	4096	256	16	1	↑ *hexadecimal point*	$\frac{1}{16}$	$\frac{1}{256}$	$\frac{1}{4096}$	

the hexadecimal radix is denoted by the subscript 16

e.g., converting the hexadecimal number 24.4_{16} to decimal:

2		4	.	4		
16^1		16^0		16^{-1}		
2×16	$+$	4×1	$+$	$4 \times \frac{1}{16}$	$= 36.25$	

hexadecimal	*decimal*
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
A	10
B	11
C	12
D	13
E	14
F	15

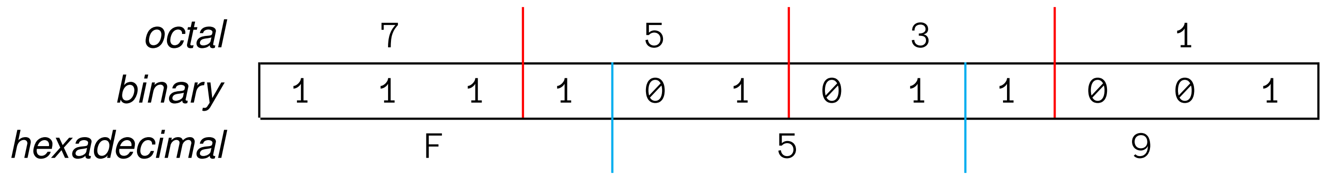
radix conversion: binary, octal and hexadecimal

one octal digit represents exactly three binary digits ($8^1 = 2^3$)

one hexadecimal digit represents four binary digits ($16^1 = 2^4$)

converting between binary and either octal or hexadecimal is easy

- octal: convert bits in groups of three
- hexadecimal: convert bits in groups of four



to convert between octal and hexadecimal

- first convert to binary

recommendation:

- keep practising conversions between binary and hexadecimal (☹)
- until you can do it *without looking at the table* (☺)

octal	*binary*	*hexadecimal*
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
	1000	8
	1001	9
	1010	A
	1011	B
	1100	C
	1101	D
	1110	E
	1111	F

radix conversion: decimal to any radix

some useful mathematical functions. . .

$\lfloor x \rfloor$ is the 'floor' operator, giving the integer part of x

$x \operatorname{div} y$ is the *integer* division operator

$x \bmod y$ is the remainder operator

$\operatorname{frac}(x)$ gives the fractional part of x

some observations. . .

moving the radix point left or right is easy

- multiplying a number by its radix moves the radix point to the right
- dividing a number by its radix moves the radix point to the left

the 'units' digit is particularly easy to 'extract' from a number

- divide an integer by its radix and the remainder is the least significant digit

radix conversion: decimal to any radix

to convert a decimal number $i.f$ to radix r

- convert one digit at a time
- convert the *integral part* and *fractional part* separately

informally:

for the integral part i

- repeatedly divide i by the radix r
- write down the remainders from *right to left*

for the fractional part f

- repeatedly multiply the fractional part f by the radix r
- write down the integer parts of the results from *left to right*

more formally...

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radix conversion: decimal to any radix

for the integral part i, convert the digits *from right to left*

repeat:

- isolate the least significant digit: $d \leftarrow i \bmod r$ (the remainder of $i \div r$)
- write down d (to the left of any previous digits)
- discard the least significant digit: $i \leftarrow i \operatorname{div} r$ (integer division operation)

until $i = 0$

example: 100.25_{10} to octal (integral part)

i	$i \bmod 8$	$i \operatorname{div} 8$
100	4	12
12	4	1
1	1	0
stop $\rightarrow 0$	$\uparrow$	
	$= \mathbf{144}_8$	

radix conversion: decimal to any radix

for the fractional part f, convert the digits *from left to right*

while $f \neq 0$, repeat:

- isolate the most significant digit: $d \leftarrow \lfloor f \times r \rfloor$ (integer part of product)
- write down d (to the right of any previous digits)
- discard the most significant digit: $f \leftarrow \mathrm{frac}\,(f \times r)$ (fractional part of product)

example: 100.25_{10} to octal (fractional part)

f	$\lfloor f \times 8 \rfloor$	$\mathrm{frac}\,(f \times 8)$
$.25$	2	$.0$
stop $\rightarrow .0$	$\uparrow$	
	$= \mathbf{.2}_8$	

$\Rightarrow 100.25_{10} = 144.2_8$

radix conversion: decimal to any radix

example: 25.25_{10} to binary

integral part

i	$i \bmod 2$	$i \operatorname{div} 2$
25	1	12
12	0	6
6	0	3
3	1	1
1	1	0
stop $\rightarrow$ 0	$\uparrow$	
	$= \mathbf{11001}_2$	

fractional part

f	$\lfloor f \times 2 \rfloor$	$\mathrm{frac}\,(f \times 2)$
.25	0	.5
.5	1	.0
stop $\rightarrow$ .0	$\uparrow$	
	$= \mathbf{.01}_2$	

$$\Rightarrow 25.25_{10} = 11001.01_2$$

summary

binary is the basis for all data representation in digital computers

- understanding binary is important

binary data is almost always represented using hexadecimal (or octal) notation

- understanding hexadecimal is important

binary data often represents numeric information

- understanding conversions to and from decimal is important

next week

arithmetic operations in positional number systems

- addition, subtraction, multiplication, division

unsigned binary arithmetic

limits of binary representation

- unsigned arithmetic overflow

glossary

base — (radix) the factor by which the significance of a digit changes from one position to the next.

digit position — the distance from a digit to the radix point, usually numbered so that integral digit positions start at 0 and increase to the left and fractional positions start at -1 and decrease to the right.

div — A mathematical operator that computes the integer quotient of two numbers.

floor ($\lfloor\ldots\rfloor$) — A mathematical operator that gives the largest integer that is no greater than the given real number.

frac — A unary mathematical operator that gives the fractional (non-integral) part of a real number.

fractional part — the part of a number to the right of the radix point, ignoring any integral part, whose value is between 0 and 1.

integral part — the part of a number to the left of the radix point, ignoring any fractional part.

least significant digit — the digit in a number (or part of a number) that has the smallest weight, usually the rightmost.

mod — a mathematical operator that gives the remainder on division of two integers.

glossary

most significant digit — the digit in a number (or part of a number) that has the largest weight, usually the leftmost.

radix — (base) the factor by which the significance of a digit changes from one position to the next.

radix point — the symbol separating the integral and fractional parts of a number. Often a dot ('.' or '·') or comma (',').

significance — the relative importance of the contribution of a digit to the overall value of a number. Significance usually increases towards the left.

weight — the amount a digit is multiplied to obtain its contribution to the overall value of a number. Weights usually increase to the left, and decrease to the right, by a factor of the radix.